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Radiation Damping of a BPS Monopole; an Implication to S-duality

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Abstract

The radiation reaction of a BPS monopole in the presence of incident electromagnetic waves as well as massless Higgs waves is analyzed classically. The reactive forces are compared to those of W boson that is interpreted as a dual partner of the BPS monopole. It is shown that the damping of acceleration is dual to each other, while in the case of finite size effects the duality is broken explicitly. Their implications on the duality are discussed.

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I. INTRODUCTION

It is a fascination that non-singular magnetic monopoles arise as classical soliton solutions in certain spontaneously broken Yang-Mills gauge theories [1,2]. These monopoles are extended objects with definite mass and couple effectively in low energies to the electromagnetic fields [3]. Of particular interest is the BPS limit of the Yang-Mills-Higgs theory [4] where one may find static multi-monopole solutions by solving the first order Bogomol'nyi equation [5]. Dynamics of the BPS monopole has received a wide attention recently in relation to S-duality (electric and magnetic duality) in supersymmetric extensions of the theory, which relates a magnetically weak coupling states of monopoles to their electric counter parts [6–10]. In N=4 supersymmetric Yang-Mills theory, this S-duality conjecture [6,7] turns out to hold exactly for the BPS saturated states [10].

The tests so far are limited to nondynamical BPS saturated states and their modular-space dynamics at large separations [10–13]. Observing that even the classical dynamics at weak coupling limits of electric or magnetic charges, comprises nontrivial dynamical excitations from the BPS saturated states, one may further test the S-duality on these excited states. Especially, the BPS monopole possesses the finite size inversely proportional to the mass of the W boson, whereas the W boson seems pointlike in the classical dynamics.

Thus one inquires whether the size effect of the BPS monopole enters explicitly in its detailed classical description. Specifically, we consider the responses of the BPS monopole to incident electromagnetic waves. The BPS monopole is expected to undergo a periodic motion and emits radiations owing to its coupling to the electromagnetic fields and the massless Higgs fields. These radiations are not included in modular space description simply because the massless fields are truncated in this approximation. In Ref. [3], it was found that ignoring the radiation reaction, the duality turns out to hold even in the presence of the radiations.

In this note, we focus on the radiation reaction of the BPS monopole. In the sense that classical motions of the BPS monopole are completely fixed by the field equations with an

asymptotic boundary condition, the problem of radiation reaction is perfectly well posed and self-contained. This is contrasted to the case of the electrodynamics, where the Abraham-Lorentz model [14] or the other attempts [15] in explaining the damping effect, are plagued with unnatural assumptions, and not sufficient in themselves [16]. We shall examine the radiation damping of the BPS monopole and compare the resulting expressions to those of the W boson.(W bosons being pointlike, we shall, anyway, follow the Abraham-Lorentz scheme in order to obtain reaction effects.) The finite size effect of the BPS monopole enters the description in consideration of higher-order corrections, and, in the following, it will be analyzed and compared to its dual part again.

In Section II, we briefly review the SU(2) Yang-Mills-Higgs theory in the BPS limit and describe the BPS monopole solutions. In addition, we present an equation describing light and Higgs-scalar scattering off a BPS monopole and its solution to the lowest order. Scattering cross-sections obtaind from this solution is none other than the dual Thomson formula, confirming the duality to this order.

In section III, we first review the previous attempts to explain the radiation damping in electromagnetism. We shall obtain the higher oder solutions to the field equation and give a description on the radiation damping and finite size effect of the BPS monopoles.

Last section comprizes the radiation reactions of the W bosons and comparisons to those of the BPS monopole. Conclusions and some comments are followed.

II. BPS MONOPOLES AND SCATTERING BY LIGHT OR HIGGS-SCALAR

The simplest field theory possessing magnetic monopoles, is the SU(2) Yang-Mills-Higgs system with the Higgs fields in the adjoint representation. We shall consider this theory in a Prasad-Sommerfield limit where the supersymmetric extension is natural. The system is described by the Lagrangian density ($a = 1, 2, 3$)

$$\mathcal{L} = -\frac{1}{4}G_a^{\mu\nu}G_{\mu\nu}^a - \frac{1}{2}(D_\mu\phi)_a(D^\mu\phi)_a \tag{2.1}$$

where

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + e\epsilon_{abc} A_b^\mu A_c^\nu, \tag{2.2}$$

$$(D_\mu \phi)_a = \partial_\mu \phi_a + e\epsilon_{abc} A_\mu^b \phi^c, \tag{2.3}$$

and the Higgs fields are subject to the asymptotic boundary condition

$$\phi_a \phi_a \to f^2 (\neq 0) \quad \text{as } r \to \infty. \tag{2.4}$$

Our metric convention is with signature $(-,+,+,+)$. The field equations read

$$(D_\mu G^{\mu\nu})_a = -e\epsilon_{abc} (D^\nu \phi)^b \phi^c, \tag{2.5}$$

$$(D_\mu D^\mu \phi)_a = 0. \tag{2.6}$$

There exist static monopole solutions satisfying the Bogomol'nyi equations

$$B_i^a = \mp (D_i \phi)^a, \qquad (B_i^a \equiv \frac{1}{2} \epsilon_{ijk} G_a^{jk}) \tag{2.7}$$

and $A_0^a = 0$. Within the spherical symmetric ansatz, a unique solution to (2.5)-(2.6) is [4]

$$A_a^i(\mathbf{r}) = \epsilon_{aij} \frac{\hat{r}_j}{er} (1 - \frac{mr}{\sinh mr}), \tag{2.8}$$

$$\phi_a(\mathbf{r}) = \pm \hat{r}_a f (\coth mr - \frac{1}{mr}) \tag{2.9}$$

where $m(\equiv ef)$ is the mass of W bosons.

To define charges, first we introduce relevant asymptotic (i.e. $r \to \infty$) fields by

$$F_{\mu\nu}^{\text{em}} = G_{\mu\nu}^a \frac{\phi^a}{|\phi|}, \quad H_\mu = -(D_\mu \phi)^a \frac{\phi^a}{|\phi|}, \tag{2.10}$$

where $F_{\mu\nu}^{\text{em}}$ and H_μ describe respectively the electromagnetic fields and the massless Higgs. The magnetic and the scalar charges are defined in a conventional way by the fluxes

$$g = \lim_{r\to\infty} \int dS^i B_{\text{em}}^i, \tag{2.11}$$

$$q_s = -\lim_{r\to\infty} \int dS^i H^i. \tag{2.12}$$

Hence the above is the BPS monopole solution with the magnetic charge, $g = \mp 4\pi/e$, the scalar charge $q_s = 4\pi/e$ and the mass (defined by evaluating the Hamiltonian) $M = 4\pi f/e$.

Static multi-monopole solutions were also constructed by solving the Bogomol'nyi equation [5]. This is possible because the magnetic forces between each monopole are balanced by the scalar forces in this BPS limit.

Note that in the broken phase, there still exists a massless U(1) field that we interpret as an electromagnetic field. The other two vector bosons become massive by Higgs mechanism. In the BPS limit, a massless Higgs is also remained and giving long range attractive interactions between the monopoles.

The question how a BPS monopole responds to asymptotic electromagnetic waves can be answered by the analysis of the field equations since all the recipes for the problem exist in the theory. Especially, in Ref. [3], the response of the BPS monopole in the presence of an asymptotic electromagnetic wave, specified by

$$\mathbf{B}^{\rm em} = \frac{M\omega^2}{g} \mathrm{Re}\left[i \left(\mathbf{a} - (\hat{\mathbf{k}} \cdot \mathbf{a})\hat{\mathbf{k}} \right) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} \right] \qquad (\omega = |\mathbf{k}|). \tag{2.13}$$

is considered. The constant real vector $\mathbf{a}$ in (2.13) will describe amplitude and direction of oscillation later. The solution to the field equations with the above asymptotic condition is constructed for the case $\omega/m \ll 1$ and $\omega a \ll 1$ to the lowest order.

We provide here a brief review on the analysis of the scattering problem in Ref. [3]. One begins the analysis by writing the ansatz for the solution

$$A^a_\mu(\mathbf{r}, t) = \bar{A}^a_\mu(\mathbf{r}) + \mathrm{Re}[\alpha^a_\mu(\mathbf{r}) e^{-i\omega t}] + O(a^2), \tag{2.14}$$

$$\phi^a(\mathbf{r}, t) = \bar{\phi}^a(\mathbf{r}) + \mathrm{Re}[\pi^a(\mathbf{r}) e^{-i\omega t}] + O(a^2) \tag{2.15}$$

where $(\bar{A}^a_i, \bar{A}^a_0 = 0, \bar{\phi}^a)$ is the static solution in (2.8)-(2.9) and (α^a_μ, π^a) are assumed to be $O(a)$. The position $\mathbf{X}$ of the monopole is defined by the zero of the field $\phi_a(\mathbf{r}, t)$, which is, of course, a gauge invariant quantity. For example, applying this definition, one may say the static monopole in (2.8)-(2.9) is located at the origin. Inserting the ansatz into the field equations (2.5)-(2.6), one finds that (α_μ, π) satisfy

$$(D_j G^{ji})_a + e\epsilon_{abc}(D^i\phi)^b\phi^c + \omega^2\alpha^i_a + i\omega(\bar{D}^i\alpha^0)_a = 0, \tag{2.16}$$

$$(\bar{D}_k\bar{D}_k\alpha^0)_a - i\omega(\bar{D}_k\alpha_k)_a + ie\omega\epsilon_{abc}\pi^b\bar{\phi}^c - e^2\epsilon_{abc}\epsilon_{bde}\alpha_0^d\bar{\phi}^e\bar{\phi}^c = 0, \tag{2.17}$$

$$(D_kD_k\phi)_a + \omega^2\pi_a + ie\omega\epsilon_{abc}\alpha_0^b\bar{\phi}^c = 0 \tag{2.18}$$

where $\bar{D}_i^{ac} \equiv \partial_i\delta_{ac} + e\epsilon_{abc}\bar{A}_i^b$ and $O(a^2)$ terms are ignored. Once the exact solutions of the above equations are found, all the relevant linear effects are included.

These equations are greatly simplified if we introduce functions $b_a^i(\mathbf{r})$ by the relation

$$G_a^{ij}(\mathbf{r},t) = \mp\epsilon_{ijk}[(D_k\phi)^a(\mathbf{r},t) + b_k^a(\mathbf{r})e^{-i\omega t} + O(a^2)]. \tag{2.19}$$

where $b_a^i(\mathbf{r})$ being $O(a)$. Using this definition for $b_a^i(\mathbf{r})$, the equations of $O(a)$ are reduced to

$$\pi^a = \frac{1}{\omega^2}\left[(\bar{D}_kb_k)_a - ie\omega\epsilon_{abc}\alpha_0^b\bar{\phi}^c\right], \tag{2.20}$$

$$\alpha_i^a = \frac{1}{\omega^2}\left[\mp\epsilon_{ijk}(\bar{D}_jb_k)^a + e\epsilon_{abc}b_i^b\bar{\phi}^c - i\omega(\bar{D}_i\alpha_0)^a\right], \tag{2.21}$$

and

$$[(\bar{D}_k\bar{D}_k + \omega^2)b_i]^a + e^2\epsilon_{abc}\epsilon_{bde}b_i^d\bar{\phi}^e\bar{\phi}^c = 0. \tag{2.22}$$

Since there is no equation for α_0, they are arbitrary functions; this implies that they are in fact pure gauge degrees of freedom. It is clear that once a solution to (2.22) is obtained, (α_i^a, π^a) are automatically given by the relations (2.20)-(2.21). In Ref. [3], the solution subject to the asymptotic condition corresponding to the incident waves in (2.13), is indeed found to the leading order:

$$b_i^a(\mathbf{r}) = \mp i\omega^2 a_i f \coth mr e^{i\mathbf{k}\cdot\mathbf{r}}\hat{r}^a \pm i\omega^2 a_i\frac{e^{i\omega r}}{er}\hat{r}^a. \tag{2.23}$$

Using (2.15), (2.20) and the definition of monopole position, one may easily show that the motion is described by

$$\mathbf{X}(t) = \mathrm{Re}[i\mathbf{a}e^{-i\omega t}] + O(a\omega)O(w/m). \tag{2.24}$$

while straightforward computations lead to expressions for the asymptotic fields in the scattering region:

$$(D_0\phi)^a(\mathbf{r},t) \sim \mp i\omega^2\hat{r}^a\left(\mathbf{a}\cdot\hat{\mathbf{k}}fe^{i\mathbf{k}\cdot\mathbf{r}-iwt} - \frac{\mathbf{a}\cdot\hat{\mathbf{r}}}{er}e^{i\omega r-iwt}\right), \tag{2.25}$$

$$(D_i\phi)^a(\mathbf{r},t) \sim \pm i\omega^2\hat{r}^a\left(\mathbf{a}\cdot\hat{\mathbf{k}}fe^{i\mathbf{k}\cdot\mathbf{r}-iwt}\hat{k}_i - \frac{\mathbf{a}\cdot\hat{\mathbf{r}}}{er}e^{i\omega r-iwt}\hat{r}_i\right), \tag{2.26}$$

$$G_a^{i0}(\mathbf{r},t) \sim i\omega^2\hat{r}^a\left[(\hat{\mathbf{k}}\times\mathbf{a})_i fe^{i\mathbf{k}\cdot\mathbf{r}-iwt} - (\hat{\mathbf{r}}\times\mathbf{a})_i\frac{e^{i\omega r-iwt}}{er}\right], \tag{2.27}$$

$$G_a^{ij}(\mathbf{r},t) \sim -i\epsilon^{ijk}\omega^2\hat{r}^a\left[(\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{a}))_k fe^{i\mathbf{k}\cdot\mathbf{r}-iwt} - (\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{a}))_k\frac{e^{i\omega r-iwt}}{er}\right]. \tag{2.28}$$

The form of incident waves are clear in the above expressions, so the generalized Lorentz force law can be checked explicitly to the order $O(a\omega)$:

$$M\ddot{\mathbf{X}}(t) = [g\mathbf{B}^{\mathrm{em}}_{\mathrm{inc}} + q_s\mathbf{H}_{\mathrm{inc}}]_{\mathbf{r}=\mathbf{X}} + O(a\omega)O(w/m), \tag{2.29}$$

where the subscript 'inc' indicates that the related quantities belong to the incident parts of the fields. From the radiation fields—the terms $O(r^{-1})$—the related differential crossections for the electromagnetic and Higgs waves are determined as

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{em}\to\mathrm{em}} = \left(\frac{g^2}{4\pi M}\right)^2\sin^2\Theta, \tag{2.30}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{em}\to\mathrm{Higgs}} = \left(\frac{g^2}{4\pi M}\right)\left(\frac{q_s^2}{4\pi M}\right)\cos^2\Theta \tag{2.31}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Higgs}\to\mathrm{em}} = \left(\frac{q_s^2}{4\pi M}\right)\left(\frac{g^2}{4\pi M}\right)\sin^2\theta, \tag{2.32}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Higgs}\to\mathrm{Higgs}} = \left(\frac{q_s^2}{4\pi M}\right)^2\cos^2\theta \tag{2.33}$$

where Θ (θ) being respectively the angle between the observation direction $\hat{\mathbf{R}}$ and $\mathbf{B}^{em}$ ($\hat{\mathbf{R}}$ and the wave vector $\mathbf{k}$). The crossections in (2.30) is the dual Thomson formula, while the other three also have dual partners in those of W bosons. As expected, the duality is still present to this order including the radiations.

Notice that there exist two parameters, $a\omega$ and ω/m for which the above perturbative scheme may be improved to their higher orders. Ignoring the terms of $O(a^2\omega^2)$ implies that one is only interested in the linear responses of the BPS monopole to the incident fields, so it is basically a weak-field approximation. We shall not improve the above treatment to this

direction because the next order equations are far more complicated than (2.16)- (2.18), and more importantly, the most we want to pursue is already in the linear responses.

Proceeding to the other direction involves simply solving the equation (2.22) to the next orders in ω/m. As we will see, one obtains radiation damping of the BPS monopole from the next order, while one finds the finite size effects of the monopole enters the force law by going one-step further. In the subsequent sections, these phenomena shall be exploited in detail.

III. RADIATION REACTION OF A BPS MONOPOLE

As is well known in classical electrodynamics, a motion of charged object subject to an external field necessarily emits radiations due to an acceleration. Since the radiation carries off energy and momentum, the subsequent motion of the object should be affected by the emission. Being this reaction force reducing the acceleration of the object, it is known as the phenomena of radiation damping of accelerations. Within the context of the classical electrodynamics, the reaction force was analyzed previously under the following assumptions: the charge distribution is rigid and the whole mass of the object arises from the electromagnetic self fields [14]. Under these assumptions, the total momentum conservation of the system leads to the following modification to the Lorentz force law; its nonrelativistic form reads [14]

$$\mathbf{F}_{\rm ext} = m_{\rm cl}\frac{d^2}{dt^2}\mathbf{X} - \frac{2}{3}\frac{q_e^2}{4\pi}\frac{d^3}{dt^3}\mathbf{X} + \sum_{n=4}^{\infty}\frac{(-1)^n}{4\pi}\frac{d^n}{dt^n}\mathbf{X}\int d\mathbf{r}d\mathbf{r}'\rho(\mathbf{r})|\mathbf{r}-\mathbf{r}'|^{n-3}\rho(\mathbf{r}') \tag{3.1}$$

where q_e being the total charge–the spatial integration of electric charge density ρ. The external force and the mass are respectively defined by

$$\mathbf{F}_{\rm ext} = \int d\mathbf{r}(\rho\mathbf{E}_{\rm ext} + \mathbf{j}\times\mathbf{B}_{\rm ext}), \tag{3.2}$$

$$m_{\rm cl} = \frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{3.3}$$

The second on the left side of (3.1) is the damping force, while the finite-size effects are described by the terms linear in the higher time derivatives of $\mathbf{X}$. For the case of point

charges, the remaining terms vanish and the radiation reaction only depends on the total charge. Partly because of the nonrelativistic nature of the expression, its form is linear in the time derivatives of the position and the external force. As noted in Ref. [16], some unsatisfactory features are present within the model. First of all, the localized distribution of nonvanishing total charges requires non-electromagnetic forces holding it stable, and even assuming the existence of such forces, the charge distribution cannot be rigid under external perturbations. Moreover, for point charges, the mass in (3.3) gives rise to infinity that might invalidate the model explaining the realistic particles.

In contrast to the Abraham-Lorentz model, the radiation reaction effects of the BPS monopole is well posed in the following sense. The mass of the BPS monopole is finite and the nonelectromagnetic holding forces are indeed present within the system; part of them comes from the attractive interaction of Higgs. The classical dynamics is totally governed by the field equation (2.5)-(2.6), i.e. it is self-contained. Considering the radiation reaction of the BPS monopole in the presence of weak incident waves, one need to find an appropriate solution to the equation in (2.22). One should note that all the linear responses of the BPS monopole to the incident waves, are included in the dynamics that is governed by the equation (2.22). Since the force law in (2.29) from the leading order solution of Eq.(2.22) does not include the reactive effects, one has to solve the higher order terms in $O(\omega/m)$ to explore the effects.

Due to the spherical symmetry of the static BPS monopole, the presence of incident plane waves with the wave vector $\mathbf{k}$, still respects the axial symmetry around the $\mathbf{k}$ axis. The most general functional form possessing the axial symmetry is

$$b_i^a(\mathbf{r}) = \mp i\omega^2 a_i f[U(r,\theta)\hat{r}^a + V(r,\theta)\hat{\theta}^a] \tag{3.4}$$

where θ being the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{k}}$. Inserting this form into the equation (2.22), one obtains

$$\left(\nabla_y^2 + \frac{\omega^2}{m^2}\right) U - \frac{2}{\sinh^2 y} U - \frac{2}{y\sinh y}(\partial_\theta V + \cot\theta V) = 0, \tag{3.5}$$

$$\left(\nabla_y^2 + \frac{\omega^2}{m^2} - 1\right) V - \frac{V}{y^2 \sin^2\theta} - 2\left(\frac{1}{\sinh^2 y} - \frac{\coth y}{y}\right) V + \frac{2}{y \sinh y}\partial_\theta U = 0. \tag{3.6}$$

where one introduces a dimensionless variable, $y \equiv mr$. When the frequency ω is smaller than the W boson mass m, the equation (3.6) tells us that V is exponentially decaying at large y. Using this large y behavior of the function V, one finds that the equations for U and V are reduced to

$$\left(\nabla_y^2 + \frac{\omega^2}{m^2}\right) U = 0, \quad V = 0 \quad (y \gg 1) \tag{3.7}$$

where only the exponentially decaying terms are suppressed. Because of the spherical symmetry of the monopole, the scattering term of a solution of the above equation also possesses the symmetry to the linear responses[1]. Having this fact in mind and the plane wave incidence, one concludes that the solution U outside the core region of the monopole should be of the form,

$$U = e^{i\mathbf{k}\cdot\mathbf{r}} - N(\omega)\frac{e^{i\omega r}}{mr} \quad (y \gg 1) \tag{3.8}$$

which may be served as an asymptotic conditions for the exact solution. We expand the functions U and V in a power series of ω/m

$$U(r,\theta) = \sum_{n=0}^{\infty}\left(\frac{\omega}{m}\right)^n U_{(n)}(r,\theta) \tag{3.9}$$

$$V(r,\theta) = \sum_{n=0}^{\infty}\left(\frac{\omega}{m}\right)^n V_{(n)}(r,\theta) \tag{3.10}$$

and solve the equation (3.5)-(3.6) order by order. Inserting the expansion (3.9)-(3.10) to (3.5)- (3.6), one finds the n^{th} order equations ($n \geq 0$) satisfy

[1] In Maxwell theory, one may illustrate this phenomena by considering, for example, the vector potential $\mathbf{A}$ for the case of spherically symmetric charge distributions with harmonic time dependence. It is given by $\mathbf{A} = \int d\mathbf{r}'\mathbf{j}(\mathbf{r}',t)\frac{e^{i\omega|\mathbf{r}-\mathbf{r}'|-i\omega t}}{|\mathbf{r}-\mathbf{r}'|} = \dot{\mathbf{X}}\frac{e^{i\omega(r-t)}}{r}\int d\mathbf{r}'\rho(r')\frac{e^{i\omega r'}}{r'} + O(\dot{X}X)$. Thus, to the linear response, the scattering solution does respect the symmetry.

$$\nabla_y^2 U_{(n)} - \frac{2}{\sinh^2 y} U_{(n)} - \frac{2}{y \sinh y} (\partial_\theta V_{(n)} + \cot\theta V_{(n)}) = -U_{(n-2)}, \tag{3.11}$$

$$(\nabla_y^2 - 1) V_{(n)} - \frac{V_{(n)}}{y^2 \sin^2\theta} - 2\left(\frac{1}{\sinh^2 y} - \frac{\coth y}{y}\right) V_{(n)} + \frac{2}{y \sinh y} \partial_\theta U_{(n)} = -V_{(n-2)} \tag{3.12}$$

where $(U_{(n)}, V_{(n)})$ for $n = -2, -1$ are introduced for convenience and simply vanish.

The generic solutions in each order comprize a particular solution together with homogeneous parts, whose coefficients in each order can be determined by comparison with the asymptotic form in (3.8). In fact the zeroth order equation is homogeneous and there exists a unique spherically symmetric nonsingular solution,

$$U_{(0)}(r) = \coth y - \frac{1}{y}, \qquad V_{(0)}(r) = 0 \tag{3.13}$$

which is consistent with the asymptotic form (3.8). For example, the other spherically symmetric solution $U_{(0)} = \coth y / y$ with vanishing $V_{(0)}$ is singular at the origin and inconsistent with the asymptotic form. This solution fixes N to be $1 + O(\omega)$, which is in good agreement with the previous analysis in (2.23). As it should be, the short distance behaviors of this zeroth order solution match with those in (2.23). The $n = 1$ equation is also homogeneous. Requiring nonsingularity at the origin and consistency with the asymptotic form uniquely fix the solution of the first order equation again, and it reads

$$U_{(1)} = iy \cos\theta \coth y - i\left(\coth y - \frac{1}{y}\right), \quad V_{(1)} = -i \frac{y \sin\theta}{\sinh y}. \tag{3.14}$$

The higher order solutions are readily solved by iterations once the zeroth and the first solutions are provided. By finding a particular solution and adding an appropriate homogeneous solution in conformity with (3.8) in a similar fashion to the lower order, one is led to a desired solution to the second order equation:

$$U_{(2)} = \frac{y}{2} - \frac{y^2}{2} \coth y \cos^2\theta - \left(\coth y - \frac{1}{y}\right) \tag{3.15}$$

$$V_{(2)} = \frac{y^2 \sin\theta \cos\theta}{2 \sinh y}. \tag{3.16}$$

When the relation $(\sin\theta)\, \hat{\theta} = -\hat{\mathbf{k}} + \hat{\mathbf{r}}(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}})$ is used, the solution $b_i^a(\mathbf{r})$ to the second order in ω/m is summarized in the expression,

$$b_i^a(\mathbf{r}) = \mp i\omega^2 a_i f \coth y \left(1 + i\omega \hat{\mathbf{k}}\cdot\hat{\mathbf{r}} + \frac{(i\omega\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})^2}{2}\right)\hat{r}_a \mp i\omega^2 a_i f \frac{2ir\omega + (ir\omega)^2\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}}{2\sinh y}(\hat{k}_a - \hat{r}_a\hat{\mathbf{r}}\cdot\hat{\mathbf{k}})$$
$$\pm i\omega^2 a_i \frac{1}{er}\left(1 + i\omega r\coth y + \frac{(ir\omega)^2}{2}\right)\left[1 - \frac{i\omega}{m} + \left(\frac{i\omega}{m}\right)^2\right]\hat{r}_a. \tag{3.17}$$

Consequently, the unknown function $N(\omega)$ is now obtaind from (3.17) by the comparison of their asymptotic forms,

$$N(\omega) = 1 - \frac{i\omega}{m} + \left(\frac{i\omega}{m}\right)^2 + O(\frac{\omega^3}{m^3}). \tag{3.18}$$

Inserting the expression (3.17) to (2.21), using the ansatz in (2.15), and evaluating the zero of the Higgs field $\phi_a(\mathbf{r}, t)$, one verifies the position is

$$\mathbf{X} = i\left\{\mathbf{a}\left[1 - \frac{i\omega}{m} + \left(\frac{i\omega}{m}\right)^2\right] - \frac{3}{2}\left(\frac{i\omega}{m}\right)^2[\mathbf{a} - \hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\mathbf{a})]\right\}e^{-i\omega t} + O(\omega^3), \tag{3.19}$$

while upon usage of the relation (2.19) and the definition (2.10), one finds the force law in terms of $\mathbf{a}$ to be

$$-i\omega^2 M\mathbf{a}e^{-i\omega t} = [g\mathbf{B}_{\text{inc}}^{\text{em}} + q_s\mathbf{H}_{\text{inc}}]_{\mathbf{r}=\mathbf{X}} \tag{3.20}$$

and in terms of the position,

$$[g\mathbf{B}_{\text{inc}}^{\text{em}} + q_s\mathbf{H}_{\text{inc}}]_{\mathbf{r}=\mathbf{X}} = M\frac{d^2}{dt^2}\mathbf{X} - \frac{g^2}{4\pi}\frac{d^3}{dt^3}\mathbf{X} + \frac{g^2}{4\pi}\frac{3}{2m}\left(\frac{d^4}{dt^4}\mathbf{X} - \hat{\mathbf{k}}\,\hat{\mathbf{k}}\cdot\frac{d^4}{dt^4}\mathbf{X}\right) + O(\omega^5). \tag{3.21}$$

When only electromagnetic fields are incident upon the BPS monopole (i.e. $\mathbf{a}\cdot\mathbf{k} = 0$), the force law reads

$$g\mathbf{B}_{\text{em}} = M\frac{d^2}{dt^2}\mathbf{X} - \frac{g^2}{4\pi}\frac{d^3}{dt^3}\mathbf{X} + \frac{g^2}{4\pi}\frac{3}{2m}\frac{d^4}{dt^4}\mathbf{X} + O(\omega^5), \tag{3.22}$$

while for the Higgs incidence alone (i.e. $\mathbf{a}\times\mathbf{k}$=0),

$$q_s\mathbf{H} = M\frac{d^2}{dt^2}\mathbf{X} - \frac{g^2}{4\pi}\frac{d^3}{dt^3}\mathbf{X} + O(\omega^5). \tag{3.23}$$

As far as the radiation dampings in (3.21), (3.22) and (3.23), are concerned, they agree with the naive expectations. Namely, the radiation damping arises from both the electromagnetic and the Higgs radiations. The electromagnetic part contributes to the force law

by $-\frac{2}{3}\frac{g^2}{4\pi}\frac{d^3}{dt^3}\mathbf{X}$ as in the case of the Abraham-Lorentz model(cf. (3.1)). Since the total energy flux of the Higgs radiation is a half of the electromagnetic flux, so does its contribution to the force law. This explains the numerical factor of the damping force in (3.21). The next order terms in the force law account for the finite size effect in the reactive forces. One sees clearly that the characteristic size relevant to the effect is none other than the size of the BPS monopole. It is interesting to note that the finite size effect is not present with the Higgs incidence alone, while the effect does exist when the electromagnetic waves are incident upon the BPS monopole. As seen in the force law (3.21), the reaction effects—specifically the finite size effect—depend explicitly upon the direction of wave incidence, i.e. the wave vector $\mathbf{k}$, which is highly contrasted to the effect of the Abraham-Lorentz model in (2.29)[2]. Reminding that the radiations off the BPS monopole possess the spherical symmetry, the directional dependence of the reactive effect is rather unusual. Presumably, the dependence is originated from the soft structures of the BPS monopole. Finally, based on the above results, the related differential crossections are found to the order ω^2/m^2:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{em}\to\mathrm{em}} = \left(\frac{g^2}{4\pi M}\right)^2\left(1-\frac{\omega^2}{m^2}\right)\sin^2\Theta, \tag{3.24}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{em}\to\mathrm{Higgs}} = \left(\frac{g^2}{4\pi M}\right)\left(\frac{q_s^2}{4\pi M}\right)\left(1-\frac{\omega^2}{m^2}\right)\cos^2\Theta \tag{3.25}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Higgs}\to\mathrm{em}} = \left(\frac{q_s^2}{4\pi M}\right)\left(\frac{g^2}{4\pi M}\right)\left(1-\frac{\omega^2}{m^2}\right)\sin^2\theta, \tag{3.26}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Higgs}\to\mathrm{Higgs}} = \left(\frac{q_s^2}{4\pi M}\right)^2\left(1-\frac{\omega^2}{m^2}\right)\cos^2\theta \tag{3.27}$$

The corrections to the dual Thomson formula vanish in case of point charges, which again reflects they arises from the finite size effects.

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[2]The explicit dependence on $\mathbf{k}$ does not present even considering the generalized Abraham-Lorentz model where the object consists of both electric and scalar charges.

IV. W BOSONS AND THEIR RADIATION REACTIONS

Though the spin contents do not match with each other, the dual partners of the BPS monopoles in the theory are known to be the W bosons. In the $N = 4$ supersymmetric Yang-Mills theory, even their spin contents exactly match [8]. However, we shall adhere to our present theory since it possesses all the essential ingredients for our purpose. In this section, we shall describe how W bosons couple to the electromagnetic fields and the Higgs fields. As we will see below explicitly, the couplings are dual to those of the BPS monopole. Namely, W bosons couple to photons and Higgs with coupling strengths e and e respectively. On the other hand, for the BPS monopole, the coupling constants of the dual photons and Higgs are respectively g and q_s $(= g)$ with the condition $eg/4\pi = 1$. The masses are again dual in the sense that the W boson mass ef is obtained from the monopole mass gf by interchanging the magnetic charge with the electric charge. To compare the dynamics of the W boson to those of the BPS monopole in the last section, we shall first describe nonrelativistic dynamics of W bosons , which will be obtained from a systematic nonrelativistic reduction of the Lagrangian (2.1). The validity of the nonrelativistic version is limited by the condition that the velocity of the W boson should be much smaller than the light velocity. Upon consideration of dynamical processes whose leading order is $O(v)$, one finds that the nonrelativistic approximation corresponds to ignoring $O(v^2)$ terms, which in turn corresponds to the linear-response approximation of the previous sections.

To begin, let us choose the unitary gauge where one may put $\phi^a = (0, 0, f + \varphi(\mathbf{r}, t))$ with a real scalar field $\varphi(\mathbf{r}, t)$. We may rewrite the Lagrange density (2.1) as

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(\mathcal{D}_\mu W_\nu - \mathcal{D}_\nu W_\mu)^\dagger(\mathcal{D}^\mu W^\nu - \mathcal{D}^\nu W^\mu) - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - m^2c^2 W^{\mu\dagger}W_\mu$$
$$+\frac{4e^2}{c^2}({W_\mu}^\dagger W_\nu - {W_\nu}^\dagger W_\mu)(W^{\mu\dagger}W^\nu - W^{\nu\dagger}W^\mu) - 2em\varphi W^{\mu\dagger}W_\mu - \frac{e^2}{c^2}\varphi^2 W^{\mu\dagger}W_\mu \tag{4.1}$$

where we rename the gauge fields with

$$A^{\rm em}_\mu = A^{(3)}_\mu, \qquad W_\mu = \frac{1}{\sqrt{2}}(A^{(1)}_\mu - iA^{(2)}_\mu), \tag{4.2}$$

and define the covariant derivative as

$$\mathcal{D}_\mu W_\nu \equiv (\partial_\mu + i\frac{e}{c}A_\mu)W_\nu. \tag{4.3}$$

Here we recover the light velocity c in order to find the nonrelativistic limit.

Note that the field equation for the W boson reads

$$\begin{aligned}&(\mathcal{D}_\nu\mathcal{D}^\nu - m^2c^2)W^\mu - \mathcal{D}^\mu\mathcal{D}^\nu W_\nu \\ &= \left[\frac{2ie}{c}F^{\mu\nu} + \frac{e^2}{c^2}(W^{\mu\dagger}W^\nu - W^{\nu\dagger}W^\mu)\right]W_\nu + \left(2em\varphi - \frac{e^2}{c^2}\varphi^2\right)W^\mu.\end{aligned} \tag{4.4}$$

Owing to the fact that the zeroth component of the above equation does not involve time derivatives, it is a constraint equation. We solve this constraint by expressing W_0 in terms of the others. In the nonrelativistic limit, one may get an explicit expression for W_0, in the $c \to \infty$ limit, as

$$W_0 = \frac{i}{mc\sqrt{2m}}e^{-imc^2t}\mathcal{D}_i\psi_i + O(c^{-2}) \tag{4.5}$$

where the field ψ_i is defined by the relation $W_i = \frac{e^{-imc^2t}}{\sqrt{2m}}\psi_i$ omitting anti-particle sector.

In terms of these variables, one finds the following desired nonrelativistic expression for action (4.1):

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + i\psi_i^\dagger\mathcal{D}_t\psi_i - \frac{1}{2m}(\mathcal{D}_i\psi_j)^\dagger\mathcal{D}_i\psi_j - e\varphi\psi_i^\dagger\psi_i - \frac{ie}{mc}B_k\epsilon^{kij}\psi_i^\dagger\psi_j \tag{4.6}$$

Here, it is clear that there are three spin degrees of freedom for the W boson and they couple to the photon as well as the massless Higgs with coupling strength (e, e). Because of the internal structure of the W boson, we have the spin-one Pauli term meaning that the W boson possesses a nonvanishing magnetic moment.

The above Lagrangian may further be reduced to a particle Lagrangian;

$$\begin{aligned}L = &\frac{1}{2}m\dot{\mathbf{X}}\cdot\dot{\mathbf{X}} + eA_0^{\mathrm{em}} + e\varphi - \frac{e}{c}\dot{\mathbf{X}}\cdot\mathbf{A}^{\mathrm{em}} + i\mathrm{tr}(Kg^{-1}\dot{g}) - \frac{e}{mc}B_kI_k \\ &- \int d\mathbf{r}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi\right)\end{aligned} \tag{4.7}$$

where $g(t)$ is the $SO(3)$ group element and the constant element K belongs to the corresponding Lie algebra. The term in the trace is the Kirillov-Kostant one-form providing symplectic structure for the classical spin variable $I_k(t)$ that is defined by

$$I_k(t)T_k \equiv g(t)Kg^{-1}(t) \tag{4.8}$$

where T_k are the three generators for the $SO(3)$ group [17]. Utilizing the symplectic structure, it can shown that the Poisson braket for the spin variable is $\{I_i, I_j\} = \epsilon_{ijk}I_k$. Thus, upon quantization, they are realized by matrices satisfying $[I_i, I_j] = i\epsilon_{ijk}I_k$. However, for our present purpose, this digression is not necessary because we shall ignore the magnetic moment interaction below. As said earlier, as far as spins are concerned, the duality is already explicitly broken in our model. Since this mismatch disappears in the $N = 4$ supersymmetric theory, we shall not further pursue its consequence below.

Though there are Higgs radiations of the W particle in addition to the electromagnetic ones, the framework of the Abraham-Lorentz model can be generalized to compute the contribution of the radiation reaction to the force law. A straightforward analysis leads to a force law:

$$[e\mathbf{E}^{\rm em}_{\rm inc} + e\mathbf{H}_{\rm inc}]_{\mathbf{r}=\mathbf{X}} = m\frac{d^2}{dt^2}\mathbf{X} - \frac{e^2}{4\pi}\frac{d^3}{dt^3}\mathbf{X}. \tag{4.9}$$

The corresponding scattering crossections of the W boson to (2.30)-(2.33) are given as the dual forms of (2.30)-(2.33), i.e. the (g, q_s) are replaced with (e, e), M with m, and the Θ angle becomes the one between the electric field and the observation direction.

The direct comparison of the force law of the monopole in (3.21) to that of W boson in (4.9) is finally at hand. The damping of accelerations are dual symmetric, while the finite size effects to the reaction, are present only in the case of the BPS monopole. In the crossections, the dual symmetry is also broken by the size effects.

A few comments are in order. First, let us observe the fact that the quantization of the modular space parameters $\mathbf{X}$, inevitably introduces the Compton size of the BPS monopole, which is the inverse of the monopole mass, i.e. $1/M$. This scale may be explicitly seen in the

Compton scattering of the BPS monopole that is basically a *photon* scattering off the BPS monopole. Therefore, one finds two scale parameters are present for dynamical processes of the BPS monopole. Namely, one is the Compton size of the monopole $1/M$ $[= 1/(gf)]$ and the other is the classical size $1/m$ $[= g/(4\pi f)]$. When the coupling between monopole and photon is weak (i.e. $g \ll 1$), the classical size is smaller than the Compton size. Nevertheless, the effect of the classical size cannot be ignored.

Considering the Compton scattering of the W boson also involves its Compton size, $1/m$. For example, the Compton scattering cross-section that is computed from the Lagrangian (2.1) to the tree level depends upon the scattered photon energy ω'

$$\omega' = \omega \left(1 + \frac{\omega}{m}(1 - \cos\theta) \right)^{-1} \tag{4.10}$$

where ω being the energy of the incident photon. Since the classical descriptions of the monopole and W-boson are dual to each other except the finite size nature of the monopole, and the effects of Compton size are entering the problem as a consequence of quantizations, one may argue that the Compton scattering of the W boson is dual to that of the monopole ignoring the finite size effect of the monopole. At least, a pair of dual scales—the Compton sizes of the monopole and W boson—is present within the theory.

On the other hand, for the case of finite size effects, naively there seems to be no scale dual to the size since the W boson is pointlike. This explains why the dual symmetry is broken in the force laws (3.21) and (4.9)[3]. The dual symmetry might be saved if one includes contributions from virtual creation of monopole–antimonopole pairs around W bosons. Since

[3] The physical situation here is different from the weak and strong coupling duality between sine-Gordon solitons and elementary excitations in massive Thirring model. Although the sine-Gordon solitons have a finite size and the Thirring excitations are elementary (pointlike), the mapping is exact for all the dynamical processes. This is possible because there is no finer excitations to probe their structures within the sine-Gordon model. On the contrary, in the case of the BPS monopole the photon can be used to probe its structures.

the energy scale involved in the process is the monopole mass, the length scale appears to be $1/M$ through ω/M. Though not clear, this size of the virtual clouds might be dual to the size of the BPS monopole[4]. Assuming the dual symmetry holds even for the finite size effect, one finds there should be a corresponding dual size effect for the motion of a W boson. This effect is completely missing in the force law (4.9), while the above consideration suggests that the virtual pair creation be the candidate for the process. Hence, upon the assumption of the dual symmetry, the contributions from the pair creation are obtained by the duality transformation from the result of the BPS monopole (e.g. (3.21)). However it should be commented that it is never clear the process of pair creation is really dual to the finite size effects of the BPS monopole, so is the dual symmetry on the finite size effect.

Another interesting feature of the Lagrangian (2.1) is that it also possesses dyonic solutions [18]. Notice that the dyons have a size proportional to $\sqrt{g^2+q^2}/(4\pi f)$, where g and q denote respectively its magnetic and electric charges. In Ref. [19], it is shown that the dyons also emit radiations when they are accelerated. (The equation similar to Eq. (2.16) that describes the response of a dyon to the incident waves, is derived in Ref. [20] from different context.) As a consequence, one expects that the damping and the finite size effects also exist for the dyons. Its detailed description and implication on the duality needs further investigations.

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[4] Even included the effect of monopole-antimonopole virtual creation, the monopole being neutral in their electric charges, still the W boson seems pointlike in their charge distribution.

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